

## Math 151 Study Guide – Test 1

### Section 2.1 – The Velocity Problem

• The one thing I want you to know best in this section is how to find the average velocity of an object on some time interval if you know how far it has gone on the interval. More than anything, I want you to realize that it's completely common sense, not something that you should have to memorize: Average velocity is simply how far you've gone divided by how much time has passed. In symbols, if  $s(t)$  is a function that tells

you where you are at a certain time  $t$ , then average velocity is given by  $\bar{v} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}$ , where  $t_2$  and  $t_1$  are

the ending time and the starting time respectively. Thinking about what this formula really says will help avoid many of the mistakes I saw on the quiz.

• I can practically guarantee you that I will ask at least one question from this section where I ask you to write a paragraph or two of explanation. Aside from the above-mentioned stuff about average velocity, the main idea of the section is how to find instantaneous velocity at a certain time: look at average velocity on increasing smaller intervals, and hopefully notice a pattern emerging. Here's the key: if the velocity does not change on a time interval, then the average velocity on that interval is exactly the same as the instantaneous velocity at the beginning. Of course, this is highly unlikely to happen in real life. But if a time interval is really small, we would expect it to be much more likely that the velocity is constant, and so average velocity becomes a better estimate of instantaneous velocity when you make the intervals real small.

• That means that to estimate instantaneous velocity at some time  $t$ , you should look at the average velocity on several intervals that include that time, and that get smaller and smaller. The pattern that comes out of the average velocities should help you guess the instantaneous velocity.

### Sample Questions:

1) If the function  $s(t) = 2t^2 - 1$  describes my position at any time, where  $s$  is in meters and  $t$  is in seconds, find my average velocity on the interval between 2 and 5 seconds. Why is this an average velocity? Next, find the average velocity on the interval between 2 and 4 seconds, 2 and 3 seconds, 2 and 2.5 seconds. Use this information to estimate the instantaneous velocity at time 2.

2) Explain why the procedure you did in part 1 allows you to **estimate** the instantaneous velocity at time 2.

### Section 2.2 – Limits

• Here's where we focused on the idea of what limits really mean – it's later that we learned methods of computing limits. So I could also ask some interpret/explain type questions from this section. Most importantly, you need to know that the study of limits has everything to do with what happens to function values as you get really close to a certain  $x$ -value, but nothing whatsoever to do with what happens when you actually get to that  $x$ .

• Make sure that you know the difference between two-sided and one-sided limits. Of course, a two-sided limit exists exactly when both one-sided limits exist and are the same.

• I think the best way to show me that you understand what limits mean is to be able to find limits by looking at the graph of a function, both one-sided and two-sided limits.

### Sample Questions:

- 1) Pg. 97, #7
- 2) Explain why it makes sense to say that  $\lim_{x \rightarrow 0} \frac{1}{x}$  does not exist.
- 3) If  $\lim_{x \rightarrow 4} f(x) = 10$ , does that mean that  $f(4) = 10$ ? Explain your answer or you stink.

### Section 2.3 – Calculating Limits

- Please note that we did this section quite differently from the book. The limit laws that they list on page 99 are important – no question about that. It's just that we didn't use them quite as extensively here: we sort of came up with our own rules that are a little easier to deal with. However, I guarantee we will need their rules a few times, so please don't just ignore them.
- Our plan for calculating limits is probably best viewed as a flow chart, which I already handed out in class. Make sure you know everything on it. (Remember that this flowchart does not work for functions defined piecewise, or some that have more than one fraction with a + or – in between. In that case you should get a common denominator and rewrite as a single fraction.)
- Please keep in mind that anytime you find a limit, it's probably a good idea to indicate to me exactly what you did to find it – don't just write down the answer. If you do, and happen to write the wrong answer, I'll have no choice other than to give you a big fat zero on the problem.
- If you have a function defined piecewise, and they want to know the limit at the place where the function breaks, you have no choice other than to lookit the one sided limits, just like it's a “#/0” problem on the flowchart.

### Sample Questions:

- 1) Find the limits using whatever method you like. If there is some work involved, like having to look at the two one-sided limits, you need to show me what you did.

$$\text{a) } \lim_{x \rightarrow 1} \frac{x-1}{\sqrt{x}-1} \quad \text{b) } \lim_{x \rightarrow -2} \frac{x+1}{x+2} \quad \text{c) } \lim_{x \rightarrow -1} \frac{2x-3x^4+3}{2x^4+x} \quad \text{d) } \lim_{y \rightarrow 3} \frac{(3+y)^2-36}{3-y}$$

### Section 2.4 - The Precise Definition of a Limit

- This is where you will impress me with your ability to handle fancy abstract technical math stuff. Of course, the idea is the same as when we first discussed limits: to say that  $\lim_{x \rightarrow a} f(x) = L$  means that you can make  $f(x)$  arbitrarily close to  $L$  by making  $x$  sufficiently close to  $a$ . But this time we say it in symbols instead of words:

For any little distance  $\varepsilon$ , you can find some corresponding little distance  $\delta$  so that choosing  $x$  within  $\delta$  of  $a$  (i.e.  $|x-a| < \delta$ ) will guarantee that  $f(x)$  is within  $\varepsilon$  of  $L$  (i.e.  $|f(x)-L| < \varepsilon$ ).

- For the most part, I'm interested in you understanding what this definition really says, and being able to explain it in your own words.
- Obviously, you need to know how to use your graphing calculator to find how close you need to make  $x$  to  $a$  to make  $f(x)$  close to the limit. There's at least a chance that I'll put a question like that on the test.

**Sample Question:** 1) In question number 6 on page 117, the correct value of  $\delta$  works out to be  $1/3$ . What exactly does that mean?

## Section 2.5 – Continuity

- By a lot, the most important thing in this section is just knowing and understanding what it means for a function to be continuous at a point - the limit exists there, the function value exists there, and those two numbers are equal. You might want to think about why this makes sense geometrically and in terms of the use of the word “continuous”.
- There’s a decent chance that I’ll ask you to decide if a function is or is not continuous at a certain  $x$ -value using the definition of continuity. That means you have to check three things: is there a point on the graph there, does the limit exist, and is the height of the point the same as the limit. If you don’t show me that you’re checking those three things, even if you get the answer right, you will get limited credit.
- We’ll need the Intermediate Value Theorem later (which is really common sense), so make sure you understand it now. I could easily ask a little short answer question to see if you understand what this Theorem really says, and why it’s common sense.
- For certain functions we don’t need a formal definition to decide if they’re continuous at a given  $x$ -value or not. (1) Polynomials are continuous for every  $x$ . (2) Rational functions (fractions where the numerator and denominator are polynomials) are continuous at any  $x$  except those that make the denominator zero.

**Sample Questions:** 1) Find all of the places where  $f(x) = 3x^2 + \frac{4x-3}{x^2-8}$  is NOT continuous.

2) Use the definition of continuity to decide whether the following function is continuous at  $x = 2$ :

$$g(x) = \begin{cases} -3x+2 & x \geq 2 \\ x-4 & x < 2 \end{cases}$$

## Section 2.6 – Limits at Infinity

• I really like this section, because these are often the kinds of limits that you need to think about a little, not just plug into a set of rules. Of course, you need to know what these type of limits mean:

$\lim_{x \rightarrow \infty} f(x) = L$  means that when  $x$  gets really, really big, all of the function values get real close to  $L$ . (More technically,  $f(x)$  gets ARBITRARILY close to  $L$  when  $x$  gets sufficiently large.)

• You ought to know, and be able to explain, why functions have horizontal asymptotes when their limit as  $x$  approaches  $\infty$  and  $-\infty$  are some finite number. Please note that a limit at infinity and a horizontal asymptote are exactly the same thing! They explain that a graph levels off way out to the right or left.

• The majority of work in this section is simply computing limits at infinity. Big important facts used in computing limits at infinity:

1)  $\lim_{x \rightarrow \infty} \frac{1}{x^r} = 0$  when  $r > 0$ .

2)  $\lim_{x \rightarrow -\infty} \frac{1}{x^r} = 0$  provided that this function is defined (i.e. no negative under a radical)

3)  $\lim_{x \rightarrow \pm\infty} x^n = \pm\infty$  for  $n > 1$ . The “ $\pm$ ” depends on whether  $n$  is positive or negative if  $n$  is a whole

number.

4) When looking at limits at infinity of polynomials, you can ignore all but the highest power terms.

**Sample Questions:** 1) Find the limits:  $\lim_{x \rightarrow \infty} 3x^2 + \frac{4}{x}$ ,  $\lim_{x \rightarrow \infty} \frac{4x^3 + 3x - 7}{3 - x - x^3}$ ,  $\lim_{x \rightarrow -\infty} \frac{5x^2 + 5x + 5}{3 - x - x^3}$

2) Find the asymptotes and use your graphing calculator to sketch the graph:

a)  $f(x) = 3x^2 + \frac{4}{x}$       b)  $g(x) = \frac{2x + 5}{x - 7}$

### **Section 2.7 – Finding Exact Velocities and Rates of Change**

- The concept here is really, really important. Finding an exact instantaneous velocity really corresponds to finding the average velocity on an interval of length zero. But you can't do this because you'd get a zero denominator. But we found that if you compute and simplify the average velocity on an arbitrary interval  $[a, a + b]$ , you get an expression with  $b$  in it. Now here's the key point, and what makes this calculus: when we have an expression for average velocity on an interval with duration  $b$ , to turn that into instantaneous velocity, we would need to squeeze the duration of that interval down to zero (as opposed to just plugging in zero for  $b$ , which we know will give us a zero denominator). Sounds just like a limit question, and it is! If we find the limit as the duration of the interval ( $b$ ) approaches zero, then we get instantaneous velocity. As far as I'm concerned, the most important thing in this section is understanding why this process should give you instantaneous velocity, and being able to compute instantaneous velocities using the procedure we outlined yesterday.

- For those that are into formulas, here's what you do to find the instantaneous velocity at time  $a$  when you're given a position function  $s(t)$ :

1) Find the average velocity on the interval  $[a, a + b]$  using the formula  $\bar{v} = \frac{s(a + b) - s(a)}{b}$ . Then simplify until you get to the point that you're able to divide out the  $b$  in the denominator. This will only happen when you either have just one term upstairs, and it has an  $b$  in it, or every term upstairs has an  $b$  in it so that you can factor out the  $b$ .

2) Find the limit as  $b$  approaches zero; assuming you simplified correctly, you can find this limit by plugging in zero for  $b$ . This tells us what the average velocity looks like on an interval of length zero, which is exactly what we mean by instantaneous velocity. (Remember that you have to SHOW me you're finding a limit by using the symbol  $\lim_{b \rightarrow 0}$  !)

- After we got good at finding the velocity at a particular time, we figured out that by just using the interval  $[t, t + b]$  where  $t$  is the time variable (as opposed to  $[a, a + b]$  where  $a$  is a specific number), we would find a function that would tell us the velocity at ANY time that we care to plug in. Much better. This leads to the definition of instantaneous velocity formula:

$$v(t) = \lim_{b \rightarrow 0} \frac{s(t + b) - s(t)}{b}$$

- Now, the big step: we know that when the original function tells us the position of some object, the limit formula above tells us the exact velocity at any time. But what if the original function describes something other than the position of an object? In that case, the fraction part of the formula measures the change in output divided by the change in input, which is the average rate of change of the quantity described by the function. When we throw in the limit, we get the instantaneous rate of change. So for any old function  $f(x)$ , the formula

$$\lim_{b \rightarrow 0} \frac{f(x+b) - f(x)}{b}$$

provides the rate at which the quantity described by the original function is changing. VERY IMPORTANT.

- Next, we found a connection between this velocity/rate of change stuff and graphs: the average velocities we used to find correspond to the slope of a line connecting two points on a position graph (i.e. a secant line), while the instantaneous velocities correspond to the slope of the tangent line at a given point (which is the line that intersects the graph at that point, and slopes in the same direction as the graph there.) So we can use our snazzy velocity formula to find the slope of the tangent line at any point on the graph. In this case, we

called it the **slope formula**, and wrote it like this:  $m(x) = \lim_{b \rightarrow 0} \frac{f(x+b) - f(x)}{b}$ .

- Finally, once you know the slope of the line tangent to the graph at a point, you can find the equation of that line, using the point-slope formula:  $y - y_0 = m(x - x_0)$ . If you're asked to do this, it's a good idea to then graph the function and your answer with your calculator to see if the line you found actually is the tangent line.

**Sample Questions:** 1) a) For the position function  $s(t) = t - 2t^2$ , compute  $v(5)$ . The units for length are feet and the units for time are seconds.

b) For the function in part (a), find the velocity function  $v(t)$ .

2) Find the equation of the line tangent to the graph of  $g(x) = \frac{x}{x-3}$  at the point with  $x$ -coordinate 5,

then graph the function and the tangent line you found.

3) The function  $r(x) = \sqrt{2x+8}$  describes the revenue made by a small business (in thousands of dollars) where  $x$  is weeks after the beginning of the year. Find the rate at which the company's revenue is changing halfway through the year. Is the revenue getting larger or smaller?