

Math 151 Sample Problems 1 Solutions

Section 2.1

1) The most important thing of course is to remember that the formula for average velocity on an interval is $\bar{v} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}$. This represents average velocity because it is simply the distance traveled divided by the amount of time that passed. This is exactly what is meant by average velocity. Easy.

$$\text{On } [2, 5], \bar{v} = \frac{s(5) - s(2)}{5 - 2} = \frac{49 - 7}{3} = 14 \text{ m./sec}$$

$$\text{On } [2, 4], \bar{v} = \frac{s(4) - s(2)}{4 - 2} = \frac{31 - 7}{2} = 12 \text{ m./sec}$$

$$\text{On } [2, 3], \bar{v} = \frac{s(3) - s(2)}{3 - 2} = \frac{17 - 7}{1} = 10 \text{ m./sec}$$

$$\text{On } [2, 2.5], \bar{v} = \frac{s(2.5) - s(2)}{2.5 - 2} = \frac{11.5 - 7}{.5} = 9 \text{ m./sec}$$

It appears that the velocity readings should level off somewhere in the neighborhood of 8, so we'll estimate that the instantaneous velocity at time 2 is 8 m/sec.

2) A good answer to this is something that you have to think about for yourself by looking at the notes, talking with others, or maybe even looking at the book. You really want to think about this, believe me – as well as how we turned average velocity into the EXACT instantaneous velocity in Section 2.7 It's easily the most important thing we've done so far.

Section 2.2

1) a) As we trace along the graph toward $t = 0$ from the left, we see that the graph goes downhill between $t = -2$ and $t = 0$, and it ends at the point $(0, -1)$. Since we only care what happens on the left here, the limit is -1 .

b) On the other hand, if you lookit what happens when you trace in along the graph from the right, the graph ends up in an open circle down at $(0, -2)$, so the limit from the right is -2 . Please note that neither one of these limits is affected in any way by the fact that the actual height of $g(t)$ at $t = 0$ is -1 : The limit only cares what happens close to $t = 0$, not at $t = 0$.

c) If you got parts a and b right, you probably got part c right also: since the two limits from the left and right do two entirely different things, this is exactly when we say that the limit does not exist.

d) As we trace along the graph toward $t = 2$ from the left, we see that the graph goes uphill between $t = 0$ and $t = 2$, and it ends in an open circle at the point $(2, 2)$. Since we only care what happens on the left here, the limit is 2.

e) On the other hand, if you lookit what happens when you trace in along the graph from the right, the graph ends up in an open circle down at $(2, 0)$, so the limit from the right is 0. Please note again that neither one of these limits is affected in any way by the fact that the actual height of $g(t)$ at $t = 2$ is $+1$: The limit only cares what happens close to $t = 2$, not at $t = 2$.

f) Again, the two one-sided limits are different, so the limit at 2 does not exist.

g) The point on the graph with first coordinate 2 is (2, 1), so $g(2) = 1$. This has NOTHING to do with any of the limit questions.

h) If you trace along the graph from both sides, getting closer to the point where $t = 4$, it's easy to see that you end up at the same place – the point (4, 3). So we would conclude that $\lim_{t \rightarrow 4} g(t) = 3$.

2) Lookit the one sided limits. It's not terribly hard to see that when you put in stuff really close to, but bigger than, zero, you get out big positive numbers. On the other hand, if you plug in x 's that are really close to, but smaller than zero, you'll get out big negative numbers. This means that we have two different candidates for what the limit could be: either $+\infty$ or $-\infty$, and no good reason to choose one or the other. So we say that the limit does not exist. Remember, our definition of limit insists that whenever we put in ANY number close to zero, we get numbers approaching the same thing. That didn't happen here.

3) Nope. The point is that limits are NOT about what happens AT a certain input, but rather CLOSE to it. The statement in the problem would be true in many cases, but it's certainly not always true.

Here's an example: for the function $f(x) = \frac{(x+6)(x-4)}{x-4}$, the limit as x approaches 4 is 10, but $f(4)$ doesn't exist.

Section 2.3

1) a) When you try to plug in 1 for x , you get 0/0. Of course, this is not the answer. When you get 0/0, this is a clue that there is some simplifying you can do. The trick here is to multiply top and bottom by the conjugate of the side that has the root:

$$\lim_{x \rightarrow 1} \frac{x-1}{\sqrt{x}-1} = \lim_{x \rightarrow 1} \frac{x-1}{\sqrt{x}-1} \cdot \frac{\sqrt{x}+1}{\sqrt{x}+1} = \lim_{x \rightarrow 1} \frac{(x-1)(\sqrt{x}+1)}{x-1} = \lim_{x \rightarrow 1} \sqrt{x}+1 = 2$$

b) When you plug in -2 for x , you get $-1/0$. Whenever this happens, there are only three choices: $+\infty$, $-\infty$, or doesn't exist. To decide which, lookit the one sided limits. When you plug in stuff a little bit less than -2 , the top will be real close to -1 , while the bottom will be a really tiny negative number. This means that the result will be a big positive number, and so we conclude that

$\lim_{x \rightarrow -2^-} \frac{x+1}{x+2} = +\infty$. On the other hand, if you plug in something a bit more than -2 , the top will still be around -1 , but the bottom will be a really tiny positive number, and so the result should be a big

negative number. So $\lim_{x \rightarrow -2^+} \frac{x+1}{x+2} = -\infty$. Since the two one-sided limits are different, the limit does

not exist. Of course, you could also have looked at the graph to get this answer. Remember that whenever plugging in gives you nonzero/zero, it means the function has a vertical asymptote there. A quick glance at the graph should tell you if both halves approach $+\infty$, both approach $-\infty$, or it's one of each (in which case the limit does not exist).

c) This is an easy one, where you can just plug in -1 for x , and get the answer:

$$\lim_{x \rightarrow -1} \frac{2x-3x^4+3}{2x^4+x} = \frac{-2-3+3}{2-1} = -2$$

d) This is another where you get 0/0 when you plug in the input that y is approaching, so we try to do some algebra:

$$\begin{aligned}\lim_{y \rightarrow 3} \frac{(3+y)^2 - 36}{3-y} &= \lim_{y \rightarrow 3} \frac{9+6y+y^2-36}{3-y} = \lim_{y \rightarrow 3} \frac{y^2+6y-27}{3-y} \\ &= \lim_{y \rightarrow 3} \frac{(y+9)(y-3)}{3-y} = \lim_{y \rightarrow 3} -(y+9) = -12\end{aligned}$$

Note that $(y-3)$ and $(3-y)$ are negatives of each other since the subtraction is in the opposite order. So when you divide that factor out, the result is -1 .

Section 2.4

1) In that question, you're illustrating the fact that $\lim_{x \rightarrow 1} \frac{2x}{x^2+4} = 0.4$ (which you can easily see since

it's an easy plug-in limit). Since they wrote the statement $\left| \frac{2x}{x^2+4} - 0.4 \right| < 0.1$, they're asking us to

make the distance between the function $\left(\frac{2x}{x^2+4} \right)$ and the limit (0.4) less than 0.1. In other words,

they want us to make the value of the function within 0.1 of the limit, so if we were to answer $\delta = 1/3$, it means that if you pick ANY x -value that lives between $2/3$ and $4/3$ (all of the numbers within $1/3$ of 1) and plug it into the function, the result will be a number that lives between 0.3 and 0.5 (that is, within 0.1 of the limit).

Section 2.5

1) First of all, notice that the " $3x^2$ " term makes no difference, since it is continuous everywhere – the only place where there can be any problems is in the denominator of the fractional part. In other words, we need to know where the denominator is zero, because this is where the function will be discontinuous.

$$x^2 - 8 = 0 \Rightarrow x^2 = 8 \Rightarrow x = \pm\sqrt{8}. \text{ So } f(x) \text{ is NOT continuous at } x = \sqrt{8} \text{ and } x = -\sqrt{8}.$$

2) The formal definition of continuity requires that you do three things to decide if f is continuous at $x = 2$: 1) Compute $f(2)$; 2) Compute $\lim_{x \rightarrow 2} f(x)$; 3) Check to see that you got the same answer.

Here we go:

- 1) $f(2) = -3(2) + 2 = -4$ since we are supposed to plug $x = 2$ into the top formula.
- 2) To compute the limit, we have to look at the two one-sided limits:

$$\lim_{x \rightarrow 2^-} f(x) = 2 - 4 = -2; \quad (\text{Plug into the part of the formula for } x < 2.)$$

$$\lim_{x \rightarrow 2^+} f(x) = -3(2) + 2; \quad (\text{Plug into the part of the formula for } x > 2.)$$

We see now that the limit does not exist, since the two one-sided limits are different. For this reason, we conclude that f can not be continuous at $x = 2$.

Section 2.6

1) a) When x goes to infinity, the first piece, $3x^2$, obviously goes to infinity. The second part, $4/x$, goes to zero since the denominator gets really big while the numerator doesn't do anything. So overall, the function goes to $\infty + 0 = +\infty$. Easy.

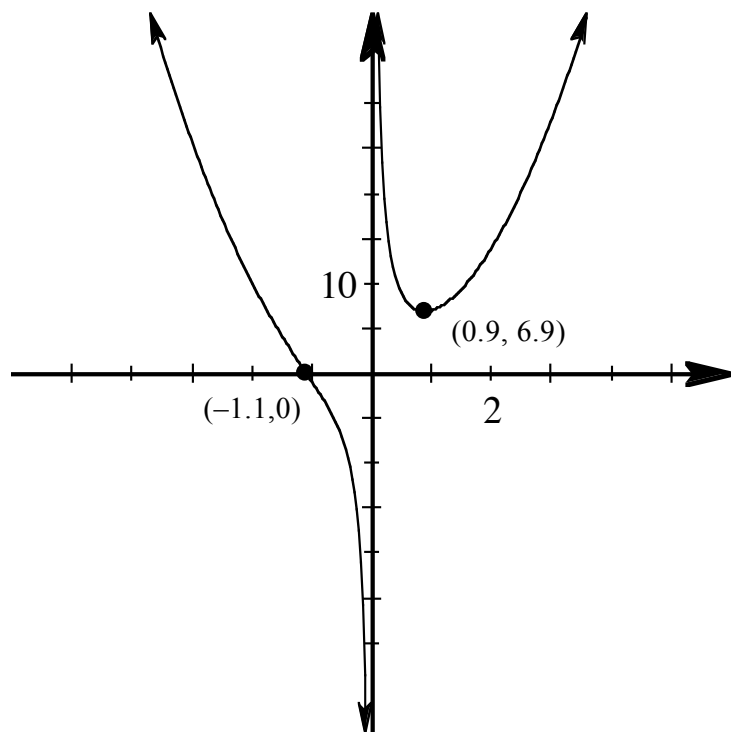
b) This is one where you have to remember that when you're looking at limits as x goes to $+\infty$ or $-\infty$, you can throw away all but the highest power terms in the numerator and denominator. (BUT ONLY WHEN x IS GOING TO \pm INFINITY!!!)

$$\lim_{x \rightarrow \infty} \frac{4x^3 + 3x - 7}{3 - x - x^3} = \lim_{x \rightarrow \infty} \frac{4x^3}{-x^3} = \lim_{x \rightarrow \infty} -4 = -4$$

c) Same game plan: throw away the lower powers of x :

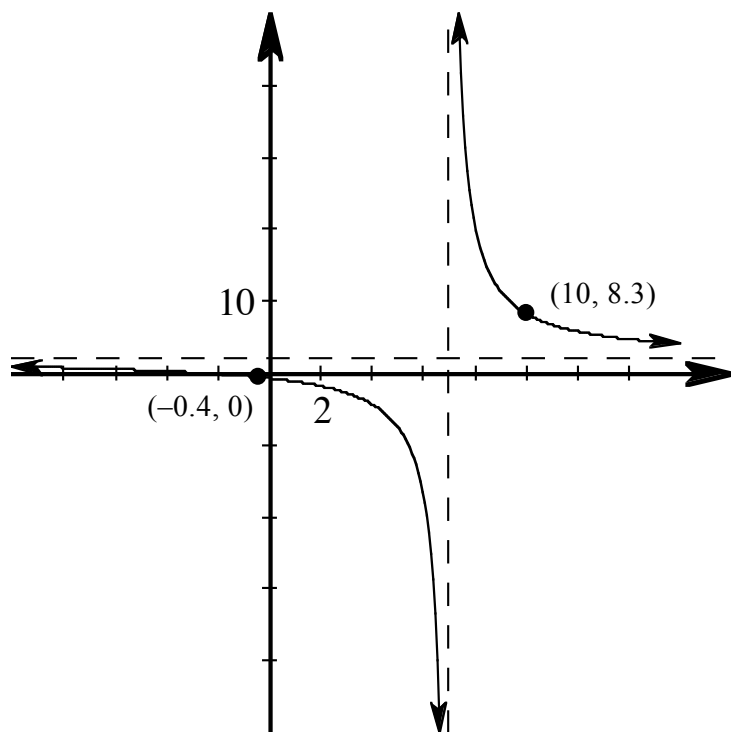
$$\lim_{x \rightarrow \infty} \frac{5x^2 + 5x + 5}{3 - x - x^3} = \lim_{x \rightarrow \infty} \frac{5x^2}{-x^3} = \lim_{x \rightarrow \infty} \frac{5}{-x} = 0$$

2) a) Since this is the same function as in 1a, and we already saw that the limit as x goes to ∞ is $+\infty$, then we already know that there is no horizontal asymptote. There is, however a vertical asymptote at $x = 0$, since this is where the denominator is zero. Here's the graph. Note the three features of a good graph that I talked about in class: a clearly labeled scale on each axis, arrows on the end to indicate that the graph continues indefinitely, and key points labeled.



b) It's pretty obvious that $x = 7$ is a vertical asymptote, again because it makes the denominator zero. To find horizontal asymptotes, we take the limits as $x \rightarrow \pm\infty$:

$$\lim_{x \rightarrow \pm\infty} \frac{2x+5}{x-7} = \lim_{x \rightarrow \pm\infty} \frac{2x}{x} = 2, \text{ so } y = 2 \text{ is a horizontal asymptote.}$$



Section 2.7

1) (a) First, find the average velocity on the interval $[5, 5+b]$:

$$\begin{aligned} \frac{s(5+b)-s(5)}{b} &= \frac{(5+b)-2(5+b)^2-(5-2(5)^2)}{b} = \frac{5+b-2(25+10b+b^2)-(-45)}{b} \\ &= \frac{5+b-50-20b-2b^2+45}{b} = \frac{-19b-2b^2}{b} = \frac{b(-19-2b)}{b} = -19-2b \end{aligned}$$

Now compute the limit as b approaches zero. Remember, this is the most important part!

$$\lim_{b \rightarrow 0} -19-2b = -19 = v(5)$$

The exact velocity at time 5 is -19 feet per second.

b) This is basically the same as the calculation above, but with the variable t in place of the number 5.

$$\begin{aligned} \frac{s(t+b)-s(t)}{b} &= \frac{(t+b)-2(t+b)^2-(t-2(t)^2)}{b} = \frac{t+b-2(t^2+2tb+b^2)-(t-2t^2)}{b} \\ &= \frac{t+b-2t^2-4tb-2b^2-t+2t^2}{b} = \frac{b-4tb-2b^2}{b} = \frac{b(1-4t-2b)}{b} = 1-4t-2b \end{aligned}$$

Again, find the limit as the duration of the interval (b) approaches zero:

$$\lim_{b \rightarrow 0} (1-4t-2b) = 1-4t = v(t)$$

The velocity function is $v(t) = 1-4t$. (Notice that if you use this formula to find $v(5)$, you get -19 , which is consistent with our answer to part a). Cool.

2) First, we'll need to find the slope function, which we get using the formula $\lim_{b \rightarrow 0} \frac{g(x+b) - g(x)}{b}$.

We'll begin by writing and simplifying the fraction part, then we'll take the limit when we're done.

$$\begin{aligned} \frac{g(x+b) - g(x)}{b} &= \frac{\frac{x+b}{x+b-3} - \frac{x}{x-3}}{b} = \frac{\frac{x+b}{x+b-3} \cdot \frac{x-3}{x-3} - \frac{x}{x-3} \cdot \frac{x+b-3}{x+b-3}}{b} \\ &= \frac{\frac{x^2 + xb - 3x - 3b}{(x+b-3)(x-3)} - \frac{x^2 + xb - 3x}{(x+b-3)(x-3)}}{b} \\ &= \frac{\frac{x^2 + xb - 3x - 3b - x^2 - xb + 3x}{(x+b-3)(x-3)}}{b} \\ &= \frac{-3b}{b(x+b-3)(x-3)} = \frac{-3}{(x+b-3)(x-3)} \end{aligned}$$

Now we compute the limit: $\lim_{b \rightarrow 0} \frac{-3}{(x+b-3)(x-3)} = \frac{-3}{(x-3)(x-3)} = \frac{-3}{(x-3)^2}$

So the slope function is $m(x) = \frac{-3}{(x-3)^2}$. Now we plug in the x -value we were given (5) to find the

slope:

$$m(5) = \frac{-3}{(5-3)^2} = -\frac{3}{4}. \text{ This is the slope of the tangent line.}$$

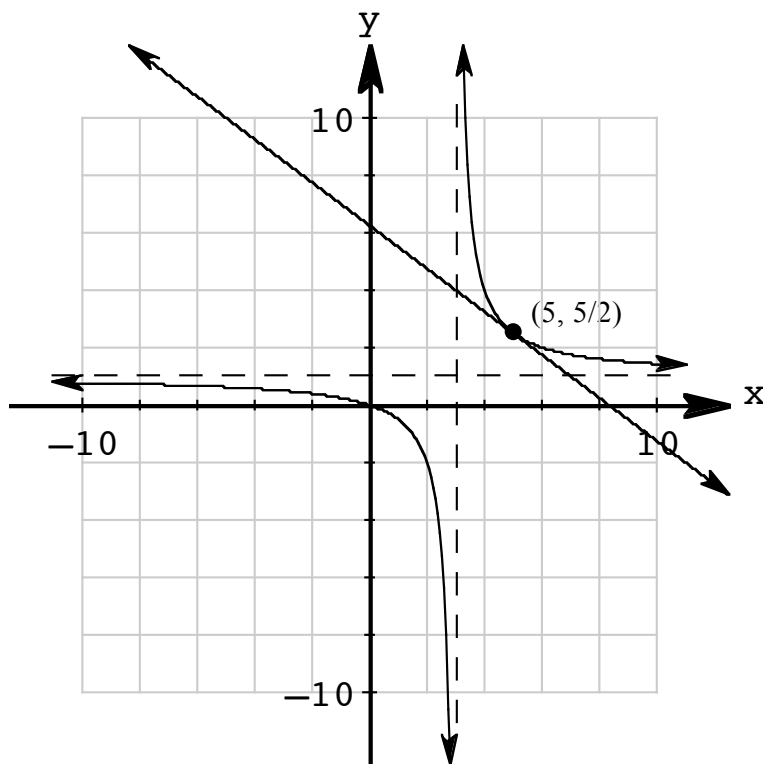
To find the point on the graph where the tangent line hits, we plug $x = 5$ back into the original function. (THINK about this – you know this is where you get points on a graph from. It has nothing to do with calculus.)

$$g(5) = \frac{5}{5-3} = \frac{5}{2}. \text{ So the point on the graph is } \left(5, \frac{5}{2}\right).$$

Finally, we use the point-slope formula to get the equation of the tangent line:

$$y - \frac{5}{2} = -\frac{3}{4}(x - 5) \Rightarrow y - \frac{5}{2} = -\frac{3}{4}x + \frac{15}{4} \Rightarrow y = -\frac{3}{4}x + \frac{25}{4}$$

Now for the graph:



Notice that the line we found appears to be the tangent line to the graph at the point with x -coordinate 5, which was, of course, the point.

3) Yet another example of using the formula we developed for instantaneous velocity, this time in a different setting.

$$\begin{aligned}
 \frac{r(x+b)-r(x)}{b} &= \frac{\sqrt{2(x+b)+8}-\sqrt{2x+8}}{b} \\
 &= \frac{\sqrt{2x+2b+8}-\sqrt{2x+8}}{b} \cdot \frac{\sqrt{2x+2b+8}+\sqrt{2x+8}}{\sqrt{2x+2b+8}+\sqrt{2x+8}} \\
 &= \frac{(2x+2b+8)-(2x+8)}{b(\sqrt{2x+2b+8}+\sqrt{2x+8})} = \frac{2b}{b(\sqrt{2x+2b+8}+\sqrt{2x+8})} \\
 &= \frac{2}{\sqrt{2x+2b+8}+\sqrt{2x+8}}
 \end{aligned}$$

Now we find the limit as b approaches zero:

$$\lim_{b \rightarrow 0} \frac{2}{\sqrt{2x+2b+8}+\sqrt{2x+8}} = \frac{2}{\sqrt{2x+8}+\sqrt{2x+8}} = \frac{2}{2\sqrt{2x+8}} = \frac{1}{\sqrt{2x+8}}$$

So now we have a formula telling us the rate at which revenue is changing. We're asked to find the rate of change halfway through the year, which is 26 weeks into the year. That means we evaluate our rate of change function for $x = 26$:

$$\frac{1}{\sqrt{2(26)+8}} = \frac{1}{\sqrt{60}} = 0.129 \text{ thousand dollars/week.}$$

The rate of change is positive, so that tells us that the revenue is increasing. The value of the rate of change tells us that revenue is increasing by \$129 per week.