Math 151 Study Guide - l'Hopital's Rule

I. The Basics

When we are trying to take limits the easy way (i.e. by simply plugging in the number that x is approaching), we have seen that occasionally we end up with 0/0. This was the case where we knew we could do some algebra to cancel out something on the bottom. The problem is that sometimes the algebra is difficult or near impossible.

Also, when we are taking limits as x approaches infinity, we often end up with ∞/∞. We came up with a routine for handling this also where you throw away lower powers of x, cancel, etc. The problem with this routine is that it only works if the numerator and denominator are both polynomials.

We call any fractional expression that goes to 0/0 or ∞/∞ when you take a limit an <u>indeterminate form</u>. We have a special rule for handling these types of problems that makes the old ways of dealing with them obsolete. (The reason we didn't learn it before is that you have to know how to compute derivatives to use the new rule, and the first time we did limits we had never even heard of derivatives.)

This shiny new rule is called "l'Hopital's Rule (or l'Hospital's Rule, depending on who you believe. For those that don't speak french, it's pronounced "Low - pee - tal"). Do the following:

1. Take the limit of the numerator and denominator first. Can you get the answer directly from this information? Or do you have an indeterminate form? If the answer appears to be 0/0 or ∞/∞, you must apply L'Hopital's rule.
2. Take the derivative of the numerator and the denominator SEPARATELY. DO NOT USE THE QUOTIENT RULE. After doing so, start over. If you still get an indeterminate form, apply L'Hopital's rule again. Keep doing so until the limit is not an indeterminate form. *IMPORTANT: If there is any simplifying that can be done after applying the rule, you must do it or things will get really screwed up*.
3. **Remember that L'Hopital's rule DOES NOT work unless you had an indeterminate form to begin with. If you use it where it does not apply, you will get the wrong answer damn near 100% of the time**.

II. Fancier Stuff

You must be aware of the fact that l'Hopital's rule is for handling indeterminate forms, and <u>only</u> for indeterminate forms. *It does not work for anything else*. The first thing you need to know is: what the hell is an indeterminate form? And where can I get one??

In doing any limit problem which has several pieces (i.e. a numerator and a denominator, two pieces multiplied together, one piece taken to a power) the standard routine is to take the limit of each piece separately, and then decide of you have a result you can handle easily, such as $0/\infty$, or an indeterminate form. Following is a list of all the indeterminate forms that you need concern yourself with; they come in six delightful flavors:

1. 0/0

2. ∞/∞
3. $0 \cdot \infty$
4. 1^{∞}
5. ∞^{0}
6. 0^{0}

Handling indeterminate forms 1 and 2 is straightforward - simply apply l'Hopital's rule. Replace both the numerator and the denominator by their respective derivatives, and try doing the limit piece by piece again. If you don't get an indeterminate form, great, you're done. If you get 0/0 or ∞/∞ again, take derivatives again, as many times as it takes to not have an indeterminate form any more. An important thing to keep in mind: after replacing numerator and denominator with their derivatives, remember that you should look for any algebraic simplification that can be done (typically, the cancelling of like terms). Often, the problem will become complicated beyond belief if you overlook some cancelling that can be done.

Indeterminate form 3 is just a bit more work. Recall that l'Hopital's rule only applies to 0/0 and ∞/∞: to apply it, we need to get the thing rewritten as a quotient. Since we have $0 \cdot \infty$, the function is written as a product. Simply take one of the pieces downstairs by making its reciprocal the denominator of a fraction. (Division is multiplication by the reciprocal of course). This will give you either a 0 / 0 or an ∞ / ∞, depending on which guy you put downstairs. Thus, there are two ways to handle a $0 \cdot \infty$; use whichever one works out to be easier. Try both!! You'll like it!!!

III. Handling Indeterminate Forms with Exponents (4-6) (Optional – you are not responsible for knowing these for the next test)

The remaining three indeterminate forms all involve exponents. Wouldn't it be nice if we had a trick to pull an exponent down and make it into a product, so that we could use the procedure for $0 \cdot \infty$? Wait a minute, we do!! Properties of logs!!!

1. Let y= the expression you are trying to take the limit of. This should be an expression that has some sort of exponent.
2. Apply ln to both sides.
3. The right side should look like ln of something to a power. By a useful property of logs, we can just swing the exponent down in front!!
4. At this point, you will have a limit that is a $0 \cdot \infty$ form. Handle this as you would any old $0 \cdot \infty$, as described above.
5. Keep in mind that when you finish doing so, you are <u>not</u> done! You have found the limit of ln y, but you were asked to find the limit of just y. So you must apply "e" to both sides to solve for y, just like we did in solving equations. Finally, this finishes the job. *The big thing to remember is that l'Hopital's rule only works for 0 / 0 and ∞ / ∞, so the whole point of handling the other 4 indeterminates is to rewrite them in one of the first two forms.*